Relationship Between the Kobayashi-Maskawa and Chau-Keung Presentations of the Quark Mixing Matrix

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Abstract

We discuss the formulas for one-to-one correspondence between the two popular parametrizations of the quark mixing matrix and the confidence limits for the mixing parameters.

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After the pioneering work by Kobayashi and Maskawa [1] different presentations for the quark mixing matrix

$$\mathbf{V} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

have been proposed (see Ref. [2] and references therein). It is however well known that no physics can depend on the parametrization of the mixing matrix and its specific form is mostly a matter of taste. It is unlikely that one can find the form of the mixing matrix best suited to all physical problems and so it is useful to have the exact formulas for one-to-one correspondence between different forms in hand. In this Brief Report we give, as a case in point, the formulas for the two popular parametrizations: the "standard" and "the" Kobayashi-Maskawa ones.

The "standard" form of the flavor mixing matrix advocated by the Particle Data Group [2] following Harari and Leurer [3], is that of Chau and Keung [4]:

$$\mathbf{V}^{(\mathrm{CK})} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix},$$

where $s_{jk} = \sin\theta_{jk}$ and $c_{jk} = \cos\theta_{jk}$ for $j, k = 1, 2, 3$; the mixing angles θ_{jk} lie in the first quadrant and $0 \le \delta_{13} < 2\pi$.

For the Kobayashi-Maskawa (KM) parametrization [1] we will use the following form [5]:

$$\mathbf{V}^{(\mathrm{KM})} = \begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ -s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix},$$

where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ for $i = 1, 2, 3$; the mixing angles θ_i lie in the first quadrant and $-\pi < \delta \le \pi$.

The charged weak currents of the standard model

$$\overline{U}\gamma_\mu \left(\frac{1+\gamma_5}{2}\right) \mathbf{V} D$$

are invariant under the gaugelike transformations of the mass eigenstates $U = (u, c, t)^T$ and $D = (d, s, b)^T$ and of the mixing matrix $\mathbf{V}$

$$U \to \exp(i\mathbf{\Omega}_U)U, \quad D \to \exp(i\mathbf{\Omega}_D)D, \tag{1a}$$

$$\mathbf{V} \to \mathbf{V}' = \exp(i\mathbf{\Omega}_U)\mathbf{V}\exp(-i\mathbf{\Omega}_D), \tag{1b}$$

with arbitrary real diagonal matrices $\mathbf{\Omega}_U$ and $\mathbf{\Omega}_D$. Thus the matrices $\mathbf{V}'$ and $\mathbf{V}$ are equivalent.

The following identities also hold true:

$$\mathrm{tr}\,(\mathbf{\Omega}_U - \mathbf{\Omega}_D) = \arg\left(\det\mathbf{V}'\det\mathbf{V}^\dagger\right), \tag{2}$$

$$V'_{\alpha i}V'_{\beta j}\left(V'_{\alpha j}V'_{\beta i}\right)^* = V_{\alpha i}V_{\beta j}\left(V_{\alpha j}V_{\beta i}\right)^*. \tag{3}$$

From Eq. (3) in particular it follows that the measure of CP violation

$$J = \mathrm{Im}\left(V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}\right) = -\mathrm{Im}\left(V_{\alpha i}V_{\beta j}V_{\gamma k}\det \mathbf{V}^\dagger\right)$$

(where the triplets α, β, γ and i, j, k are arbitrary cyclic permutations of the u, c, t and d, s, b, respectively) is invariant under the transformation (1b).

The substitution $\mathbf{V} = \mathbf{V}^{(\mathrm{KM})}$ and $\mathbf{V}' = \mathbf{V}^{(\mathrm{CK})}$ gives the sought relationship. Using the identities $|V^{(\mathrm{CK})}_{\alpha i}| = |V^{(\mathrm{KM})}_{\alpha i}|$, it is a simple matter to derive the relations between the mixing angles (cf. [6]):

$$s_{12} = \frac{s_1c_3}{\sqrt{1-s_1^2s_3^2}}, \qquad s_{13} = s_1s_3,$$
$$s_{23} = \sqrt{\frac{c_1^2c_2^2s_3^2 + s_2^2c_3^2 + 2c_1s_2c_2s_3c_3\cos\delta}{1-s_1^2s_3^2}},$$

and from Eqs. (1b) and (3), we find the relationship between the CP violating phases δ_{13} and δ:

$$\sin\delta_{13} = \frac{c_2s_2(1-s_1^2s_3^2)\sin\delta}{\Lambda},$$
$$\cos\delta_{13} = \frac{\Lambda_0}{\Lambda}\cos^2\frac{\delta}{2} + \frac{\Lambda_\pi}{\Lambda}\sin^2\frac{\delta}{2}.$$

Here

$$\begin{aligned}
\Lambda \equiv \Lambda(\theta_1,\theta_2,\theta_3;\delta) &= (1-s_1^2s_3^2)c_{23}s_{23} \\
&= \sqrt{\Lambda_0^2\cos^2\frac{\delta}{2} + \Lambda_\pi^2\sin^2\frac{\delta}{2} + 4c_1^2s_2^2c_2^2s_3^2c_3^2\sin^2\delta}, \\
\Lambda_0 &= (c_2c_3 - c_1s_2s_3)(c_1c_2s_3 + s_2c_3), \\
\Lambda_\pi &= (c_2c_3 + c_1s_2s_3)(c_1c_2s_3 - s_2c_3),
\end{aligned}$$

and it is anticipated that $\Lambda \neq 0$ in conformity with the experimental limits [7]. Clearly $\Lambda = |\Lambda_\delta|$ at $\delta = 0, \pi$.

The implicit form of the phase matrices $\mathbf{\Omega}_U$ and $\mathbf{\Omega}_D$ is defined modulo a common matrix $c \cdot \mathrm{diag}(1,1,1)$ with c an arbitrary constant. This constant may be chosen so that the phases of the fields u and d do not change under the transformation (1a). By direct substitution, one can verify that

$$\mathbf{\Omega}_U = \mathrm{diag}\left(0,\ \frac{\delta_{13}-\delta+\Omega}{2},\ \frac{\delta_{13}-\delta+2\pi-\Omega}{2}\right),$$
$$\mathbf{\Omega}_D = \mathrm{diag}\left(0,\ 0,\ \delta_{13}\right),$$

where the angle Ω is determined by

$$\sin\Omega = \frac{c_1s_3c_3\sin\delta}{\Lambda},$$
$$\cos\Omega = \frac{\Lambda_0}{\Lambda}\cos^2\frac{\delta}{2} - \frac{\Lambda_\pi}{\Lambda}\sin^2\frac{\delta}{2}.$$

It is easy to verify that $\sin\delta_{13} = \sin\Omega = 0$ and $\cos\delta_{13} = (-1)^n \cos\Omega = \mathrm{sign}\,(\Lambda_\delta)$ at $\delta = n\pi$, $n = 0, 1$. Or put in another way,

$$\delta_{13}\Big|_{\delta=0} = \Omega\Big|_{\delta=0} = \begin{cases} 0, & \text{if } \; c_2c_3 > c_1s_2s_3, \\ \pi, & \text{if } \; c_2c_3 < c_1s_2s_3, \end{cases}$$

and

$$\delta_{13}\Big|_{\delta=\pi} = \pi - \Omega\Big|_{\delta=\pi} = \begin{cases} 0, & \text{if } \; s_2c_3 < c_1c_2s_3, \\ \pi, & \text{if } \; s_2c_3 > c_1c_2s_3. \end{cases}$$

The formulas for the inverse transformation may be obtained from the foregoing ones by interchanging

$$\begin{gathered} s_1 \leftrightarrow c_{13}, \quad s_2 \leftrightarrow s_{23}, \quad s_3 \leftrightarrow c_{12}, \\ c_1 \leftrightarrow s_{13}, \quad c_2 \leftrightarrow c_{23}, \quad c_3 \leftrightarrow s_{12}, \\ \sin\delta \leftrightarrow \sin\delta_{13}, \quad \cos\delta \leftrightarrow -\cos\delta_{13}. \end{gathered}$$

With the derived formulas we can obtain the 90% confidence limits for the KM mixing angles using the experimental constraints on the CK mixing angles and the constraints on the magnitude of the elements $V_{\alpha i}$ imposed by unitarity [2]. The angle θ_1 lies in the narrow interval $12.6^\circ \div 12.9^\circ$ ($s_1 = 0.218$ to 0.224) while the confidence interval for the angle θ_3 proves to be comparatively wide: $0.512^\circ \div 1.31^\circ$ ($s_3 = 0.009$ to 0.023). Fig. 1 shows the angle θ_2 vs δ_{13}. The maximum uncertainty in this angle is about 1.6° at $\delta_{13} \sim 160^\circ - 200^\circ$.

In Fig. 2 we show the function δ vs δ_{13} for the same range of the CK mixing angles. One can see that with the modern 90% confidence limits on the $|V_{\alpha i}|$, the maximum uncertainty in prediction of the KM phase δ at a given value of the CK phase δ_{13} is about 34°. Fig. 2 demonstrates that the current 90% limits on the mixing angles present unambiguous correspondence: $\delta = 0$ at $\delta_{13} = 0$ and $\delta = \pi$ (or $-\pi$, that is the same) at $\delta_{13} = \pi$. It is also seen that the maximal CP nonconservation in the standard presentation ($\delta_{13} = \pi/2$ or $3\pi/2$) leads to δ different from $\pm\pi/2$ [8]. Namely at $\delta_{13} = \pi/2$ the phase δ lies in the interval $1.75 \div 2.18$ ($100.3^\circ \div 124.8^\circ$). The measure of CP nonconservation, J, is of course independent on the parametrization:

$$\begin{aligned} J &= \frac{1}{8}\sin\delta\sin\theta_1 \prod_i \sin 2\theta_i \\ &= \frac{1}{8}\sin\delta_{13}\cos\theta_{13} \prod_{j<k} \sin 2\theta_{jk} \\ &= (1.36 \div 5.23) \times 10^{-5} \sin\delta_{13}. \end{aligned}$$

Summarizing we derived the exact formulas connecting the KM and CK (standard) presentations of the mixing matrix which may be helpful for the study of the CP violation in quark sector.

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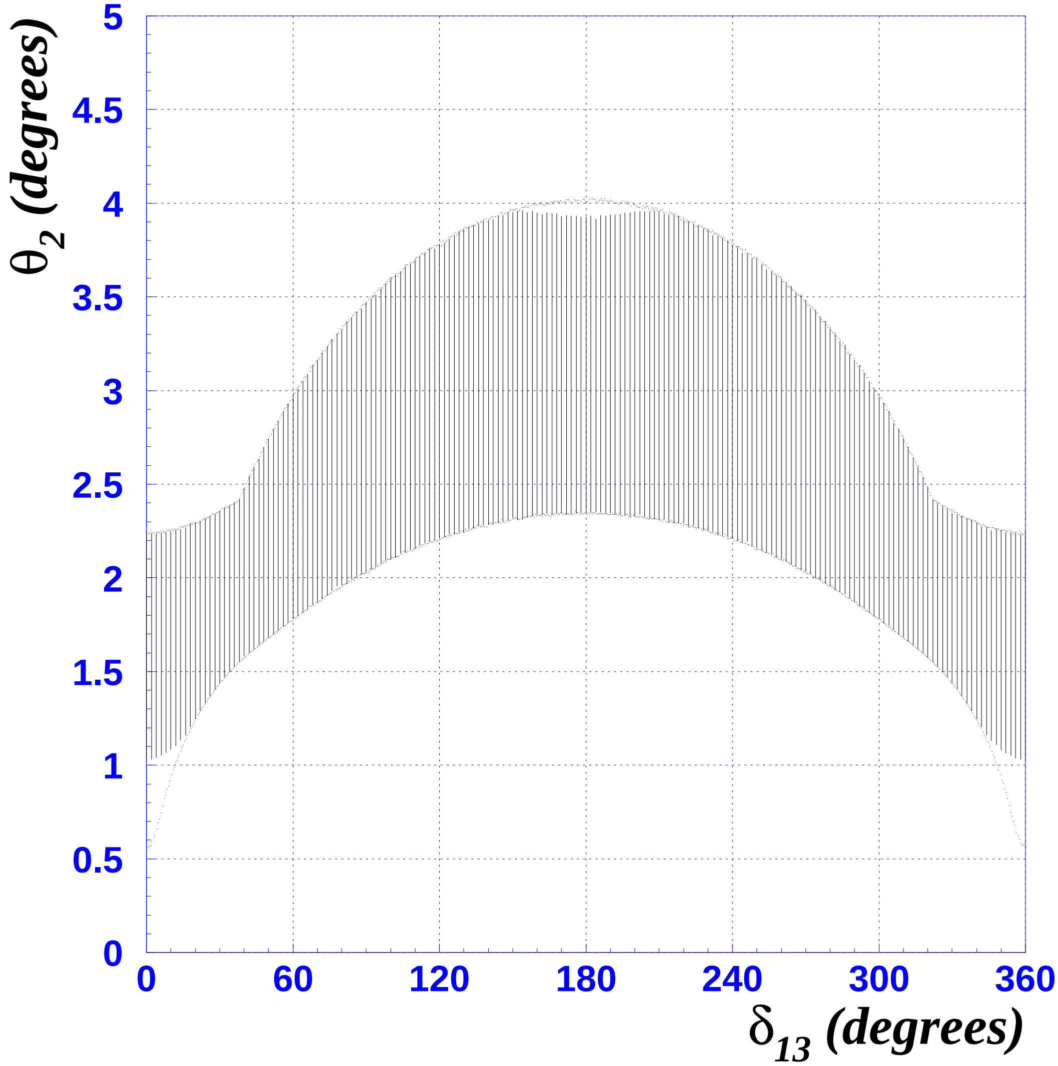

FIG. 1. θ_2 vs δ_{13} for the CK mixing angles θ_{jk} from the 90% confidence intervals given by $s_{12} = 0.218$ to 0.224, $s_{23} = 0.032$ to 0.048, and $s_{13} = 0.002$ to 0.005 [2]. The shaded area satisfies the unitarity constraints.

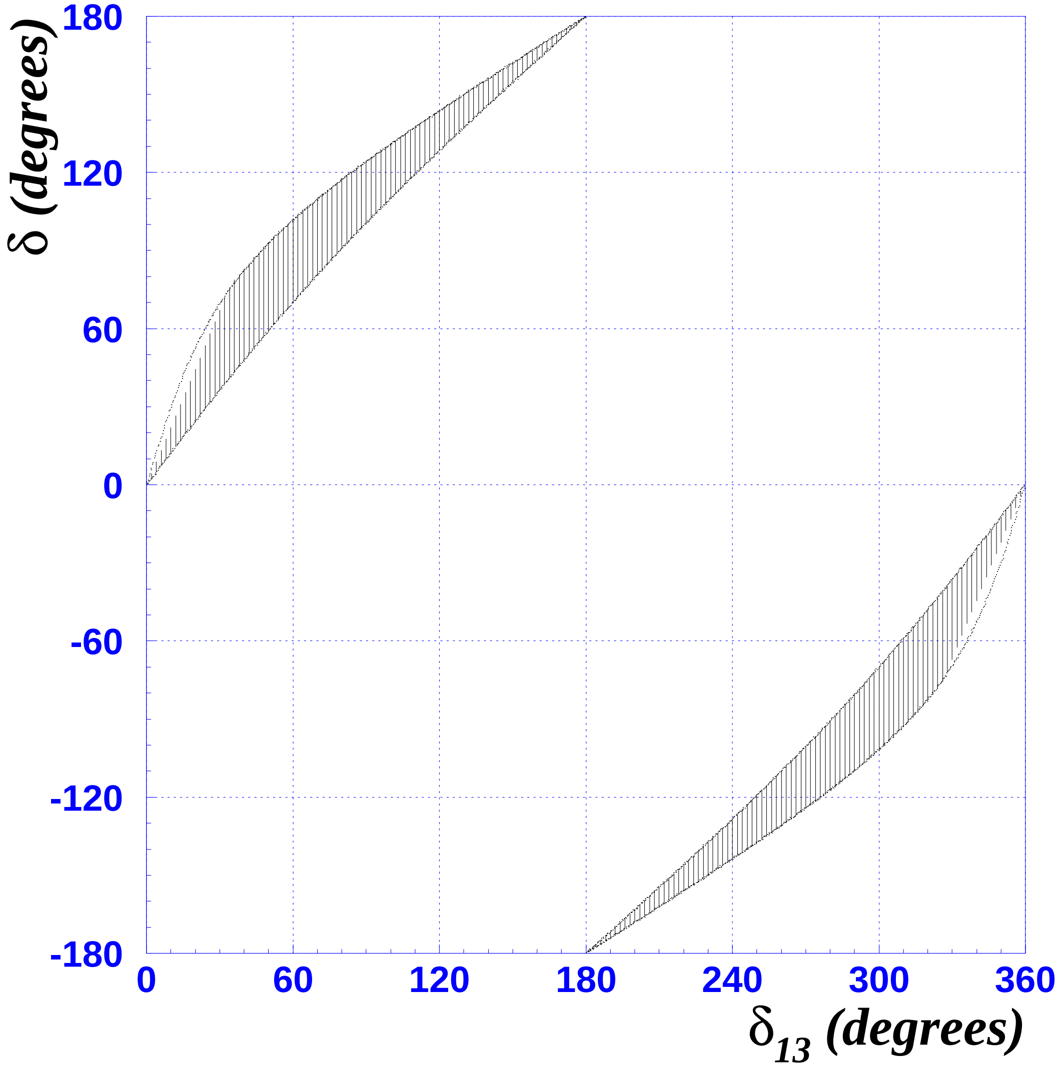

FIG. 2. δ vs δ_{13} for the same intervals of the CK mixing angles as in Fig. 1. The shaded areas satisfy the unitarity constraints.